Competition Graphs of Jaco Graphs and the Introduction of the Grog Number of a Simple Connected Graph

(Johan Kok, Susanth C, Sunny Joseph Kalayathankal)[1]

Abstract

Let $G^{\rightarrow}$ be a simple connected directed graph on $n \geq 2$ vertices and let V^* be a non-empty subset of $V(G^{\rightarrow})$ and denote the undirected subgraph induced by V^* by, $\langle V^* \rangle$. We show that the *competition graph* of the Jaco graph $J_n(1), n \in \mathbb{N}, n \geq 5$, denoted by $C(J_n(1))$ is given by:

$$C(J_n(1)) = \langle V^* \rangle_{V^*=\{v_i | 3 \leq i \leq n-1\}} - \{v_i v_{m_i} | m_i = i + d^+_{J_n(1)}(v_i), 3 \leq i \leq n-2\} \cup \{v_1, v_2, v_n\}.$$

Further to the above, the concept of the *grog number* $g(G^{\rightarrow})$ of a simple connected directed graph $G^{\rightarrow}$ on $n \geq 2$ vertices as well as the general *grog number* of underlying graph G, will be introduced. The *grog number* measures the efficiency of an *optimal predator-prey strategy* if the simple directed graph models a ecological predator-prey web.

We also pose four open problems for exploratory research.

Keywords: Jaco graph, Competition graph, Grog number

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1 Introduction

For a general reference to notation and concepts of graph theory see [1]. For ease of self-containess we shall briefly introduce the concept of a *competition graph.*

1.1 The Competition Graph of a simple connected directed graph $G^{\rightarrow}$

The concept of the competition graph $C(G^{\rightarrow})$ of a simple connected directed graph $G^{\rightarrow}$ on $n \geq 2$ vertices, was introduced by Joel Cohen in 1968 [2]. Much research has followed and recommended reading can be found in [4, 5, 6 *together with all their references*]. The concept of competition graphs found application in amongst others, Coding theory, Channel allocation in communication, Information transmission, Complex systems modelled in energy and economic applications, Decionmaking based mainly on opinion influences and Predator-Prey dynamical systems.

For a simple connected directed graph $G^{\rightarrow}$ with vertex set $V(G^{\rightarrow})$ the competition graph $C(G^{\rightarrow})$ is the simple graph *(undirected and possibly disconnected)* having $V(C(G^{\rightarrow})) = V(G^{\rightarrow})$ and the edges $E(C(G^{\rightarrow})) = \{vy|$ *if at least one vertex* $w \in V(G^{\rightarrow})$ *exists such that the arcs* $(v, w), (y, w)$ *exist*$\}$.

Let $G^{\rightarrow}$ be a simple connected directed graph and let V^* be a non-empty subset of $V(G^{\rightarrow})$ and denote the undirected subgraph induced by V^* by, $\langle V^* \rangle$.

1.2 The Competition graph of the Jaco graph, $J_n(1), n \in \mathbb{N}$

For ease of reference the definition and basic properties of the Jaco graph $J_n(1), n \in \mathbb{N}$ will be repeated.

The infinite Jaco graph (*order 1*) was introduced in [3], and defined by $V(J_\infty(1)) = \{v_i|i \in \mathbb{N}\}$, $E(J_\infty(1)) \subseteq \{(v_i, v_j)|i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in E(J_\infty(1))$ if and only if $2i - d^-(v_i) \geq j$.

The graph has four fundamental properties which are; $V(J_\infty(1)) = \{v_i|i \in \mathbb{N}\}$ and, if v_j is the head of an edge (arc) then the tail is always a vertex $v_i, i < j$ and, if v_k, for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices $v_\ell, k < \ell < j$ are tails of arcs to v_j and finally, the

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degree of vertex k is $d(v_k) = k$. The family of finite directed graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and edges arcing to vertices) $v_t, t > n$. Hence, trivially we have $d(v_i) \leq i$ for $i \in \mathbb{N}$. It is important to note that the general definition of a finite Jaco graph [3], prescribes a well-defined orientation. So we have one well-defined orientation of the $2^{\epsilon(J_n(1))}$ possible orientations.

Theorem 1.1. *For the Jaco graph $J_n(1), n \in \mathbb{N}, n \geq 5$, the competition graph $C(J_n(1))$ is given by:*

$$C(J_n(1)) = \langle V^* \rangle_{V^* = \{v_i | 3 \leq i \leq n-1\}} - \{v_i v_{m_i} | m_i = i + d^+_{J_n(1)}(v_i), 3 \leq i \leq n-2\} \cup \{v_1, v_2, v_n\}.$$

Proof. The well-defined orientation of Jaco graphs renders the competition graphs of $J_n(1)_{1 \leq n \leq 4}$ to be isolated vertices only.

From the definition of a Jaco graph it follows clearly that $J_5(1)$ is the smallest Jaco graph for which the vertex v_5 exists such that arcs (v_3, v_5) and (v_4, v_5) exist to allow the edge $v_3 v_4$ in the *competition graph*. It follows easily that vertices v_1, v_2 are isolated vertices in $C(J_5(1))$.

Since $d^+_{J_5(1)}(v_5) = 0$, vertex v_5 remains isolated in $C(J_5(1))$. Hence, the edges *(previously arcs respectively)*, $v_3 v_5$ and $v_4 v_5$ do not exist in $C(J_5(1))$. It implies that the edge *(previously an arc)*, $v_3 v_5 = v_3 v_{m_3}, m_3 = 3 + d^+_{J_5(1)}(v_3), 3 \leq 3 \leq 3 = 5 - 2$ does not exist. So the result:

$$C(J_n(1)) = \langle V^* \rangle_{V^* = \{v_i | 3 \leq i \leq n-1\}} - \{v_i v_{m_i} | m_i = i + d^+_{J_n(1)}(v_i), 3 \leq i \leq n-2\} \cup \{v_1, v_2, v_n\},$$
holds for $n = 5$.

We will now apply induction. Assume the result holds for $n = k$ hence, assume:

$$C(J_k(1)) = \langle V^* \rangle_{V^* = \{v_i | 3 \leq i \leq k-1\}} - \{v_i v_{m_i} | m_i = i + d^+_{J_k(1)}(v_i), 3 \leq i \leq k-2\} \cup \{v_1, v_2, v_k\}.$$
Also assume $J_k(1)$ has Jaconian vertex v_i.

Consider $n = k+1$. This extension adds the vertex v_{k+1} and the set of arcs, $\{(v_{i+1}, v_{k+1}), (v_{i+2}, v_{k+1}), (v_{i+3}, v_{k+1}), ..., (v_k, v_{k+1})\}$ to $J_k(1)$ to obtain $J_{k+1}(1)$. From the definition of the *competition graph* it follows that edges $v_{i+1} v_k, v_{i+2} v_k, ..., v_{k-1} v_k$ are defined in $C(J_{k+1}(1))$. Since, $k - 1 = (k + 1) - 2$ we have the edges $v_{i+1} v_k, v_{i+2} v_k, ..., v_{(k+1)-2} v_k$ defined in $C(J_{k+1}(1))$.

Clearly vertices v_1 and v_2 remain isolated vertices in $C(J_{k+1}(1))$. Equally evident is that $d^+_{J_{k+1}(1)}(v_{k+1}) = 0$, so v_{k+1} is an isolated vertex in $C(J_{k+1}(1))$. Hence the result:

$$C(J_{k+1}(1)) = \langle V^* \rangle_{V^* = \{v_i | 3 \leq i \leq (k+1)-1\}} - \{v_i, v_{m_i} | m_i = i + d^+_{J_{k+1}(1)}(v_i), 3 \leq i \leq (k+1) -$$

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$2\} \cup \{v_1, v_2, v_{k+1}\}$, holds.

Thus the result:

$$C(J_n(1))_{n \geq 5} = \langle V^* \rangle_{V^* = \{v_i | 3 \leq i \leq n-1\}} - \{v_i v_{m_i} | m_i = i + d^+_{J_n(1)}(v_i), 3 \leq i \leq n-2\} \cup \{v_1, v_2, v_n\}$$

is settled through induction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2 Grog Numbers of Simple Connected Directed Graphs

For a simple connected graph G on $n \geq 2$ vertices we consider any orientation $G^\rightarrow$ thereof. Label the vertices randomly $v_1, v_2, v_3, ..., v_n$. The aforesaid vertex labelling is called is called *indicing* and a specific labelling pattern is called an *indice* of G. Consider the graph to represent a predator-prey web. A vertex v with $d^+_{G\rightarrow}(v) = 0$ is exclusively *prey*. To the contrary a vertex w with $d^-_{G\rightarrow}(w) = 0$ is exclusively *predator*. A vertex z with $d_{G\rightarrow}(z) = d^+_{G\rightarrow}(z)_{>0} + d^-_{G\rightarrow}(z)_{>0}$ is a mix of *predator-prey*.

Let a vertex labelled v_i have an initial *predator$_{\geq 0}$-prey$_{\geq 0}$ population* of exactly $\rho(v_i) = i$. So generally there is no necessary relationship between the initial *predator$_{\geq 0}$-prey$_{\geq 0}$ population* $\rho(v_i) = i$ and $d_{G\rightarrow}(v_i) = d^+_{G\rightarrow}(v_i)_{\geq 0} + d^-_{G\rightarrow}(v_i)_{\geq 0}$.

2.1 The Grog algorithm

The *predator-prey* dynamics now follow the following rules.

Grog algorithm:[2]
0. Consider the initial graph $G^\rightarrow$.
1. Choose any vertex v_i and predator along any number $1 \leq \ell \leq d^+_{G\rightarrow}(v_i) \leq i$ of out-arcs or along any number $1 \leq \ell \leq i < d^+_{G\rightarrow}(v_i)$ of out-arcs, with only one predator per out-arc provided that the preyed upon vertex v_j has $j \geq 1$.
2. Remove the out-arcs along which were predatored and set $d^+_*(v_i) = d^+_{G\rightarrow}(v_i) - \ell$, and for all vertices $v_{j \neq i}$ which fell prey, set $d^-_*(v_j) = d^-_{G\rightarrow}(v_j) - 1$.
3. Set the *predator$_{\geq 0}$-prey$_{\geq 0}$ populations* $\rho_*(v_i) = i - \ell$ and $\rho_*(v_{j \neq i}) = j - 1$.

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[2]Admittedly, the Grog algorithm has been described informally. See Open problem 3.

4. Consider the next amended graph $G_*^{\rightarrow}$ and apply rules 1, 2, 3 and 4 thereto if possible. If not possible, exit.

Observation 1: We observe that since both the $predator_{\geq 0}$-$prey_{\geq 0}$ *population* of all vertices, and the number of out-arcs embedded in $G^{\rightarrow}$ as well as those respectively found in the iterative amended graphs $G_*^{\rightarrow}$ are finite, the Grog algorithm will always terminate. So it can be said informally that the Grog algorithm is well-defined.

We note that rule 1 allows us to choose any vertex v_i per iteration. Following that, any number of the existing out-arcs from v_i can be chosen to predator along. Collectively, the specific iterative choices will be called the *predator-prey strategy*. Generally, a number of *predator-prey strategies* may exist for a given $G^{\rightarrow}$ and the set of all possible strategies is denoted, $S(G^{\rightarrow})$. Amongst the strategies there will be those who for a chosen vertex v_i, consecutively predator along the maximal number of out-arcs available at v_i. These strategies are called *greedy strategies* and a greedy strategy $s_k \in S(G^{\rightarrow})$ is denoted gs_k.

Observation 2: We observe that if a *predator-prey strategy* $s_k \in S(G^{\rightarrow})$ is repeated for a specific graph $G^{\rightarrow}$, the amended graph $G_*^{\rightarrow}$ found on termination (exit step) is unique. Put differently, we informally say the *predator-prey strategy* s_k is well-defined.

In the final amended graph $G_*^{\rightarrow}$ (exit step) we will find each vertex v_i has $\rho_{G_*^{\rightarrow}}(v_i) \geq 0$. Some residual arcs may be present as well.

Lemma 2.1. *In the final amended graph $G_*^{\rightarrow}$ (exit step) there will be at least one vertex v_i and at least one vertex v_j with $\rho_{G_*^{\rightarrow}}(v_i) = 0$ and $\rho_{G_*^{\rightarrow}}(v_j) > 0$.*

Proof. Part 1: Since G is a simple connected graph the open neighborhood of v_1 has $N(v_1) \neq \emptyset$. If in the final amended graph $G_*^{\rightarrow}$ (exit step), we have $\rho_{G_*^{\rightarrow}}(v_1) = 0$, Part 1 of the result holds. If $\rho_{G_*^{\rightarrow}}(v_1) \neq 0$, it implies that $\rho_{G_*^{\rightarrow}}(v_k) = 0$, $\forall v_k \in N(v_1)$. Hence, Part 1 of the result holds.

Part 2: Since $G^{\rightarrow}$ is a a simple connected directed graph the extremal case is that v_n is preyed upon or predator on, cumulatively over all other $n - 1$ vertices of $G^{\rightarrow}$. Thus we have $\rho_{G_*^{\rightarrow}}(v_n) = \rho_{G^{\rightarrow}}(v_n) - (n-1) = n - (n-1) = 1 > 0$. Hence, Part 2 of the result holds. $\square$

Definition 2.1. *For a predator-prey strategy s_k and the final amended graph $G_*^{\rightarrow}$ (exit step), the cumulative residual, $predator_{\geq 0}$-$prey_{\geq 0}$ population over all vertices is denoted and defined to be $r_{s_k}(G^{\rightarrow}) = \sum\limits_{\forall v_i} \rho_{G_*^{\rightarrow}}(v_i)$.*

Definition 2.2. *The grog number of $G^{\rightarrow}$ is defined to be $g(G^{\rightarrow}) = min(r_{s_k}(G^{\rightarrow}))_{\forall s_k \in S(G^{\rightarrow})}$ or equivalently, $g(G^{\rightarrow}) = min(r_{gs_k}(G^{\rightarrow}))_{\forall gs_k \in S(G^{\rightarrow})}$.*

Definition 2.3. *The grog number of a simple connected graph G is defined to be $g(G) = min(g(G^{\rightarrow}))$ over all possible orientations of G.*

Consider a simple connected graph G on $n \geq 2$ vertices with $\epsilon(G)$ edges. It is easy to see that the n vertices can be randomly labelled *(indiced)*, through $v_1, v_2, v_3, ..., v_n$ in $n!$ ways. Equally easy to see that the edges can be orientated in $2^{\epsilon(G)}$ ways. Hence, $\frac{1}{2}n!.2^{\epsilon(G)}$ distinct predator-prey webs can be constructed from $G^{\rightarrow}$.

Let $\mathbb{P}_{s_k}(G^{\rightarrow}) = \{(v_i \leadsto v_j) | v_i$ is predator to $v_j\}$. Call an arc $(v_i \leadsto v_j) \in \mathbb{P}_{s_k}(G^{\rightarrow})$ a predator arc. Denote the cardinality of $\mathbb{P}_{s_k}(G^{\rightarrow})$ by $c(\mathbb{P}_{s_k}(G^{\rightarrow}))$.

From the Grog algorithm the interative sequence of s_k can be recorded as an ordered string. So if s_k terminates (exit step) after t iterations we can express s_k as,
$s_k = ((v_{i_1} \leadsto v_{j_1}), (v_{i_2} \leadsto v_{j_2}), (v_{i_3} \leadsto v_{j_3}), ..., (v_{i_t} \leadsto v_{j_t}))$. Clearly any pair of predator arcs say, (v_{i_ℓ}, v_{j_ℓ}) and (v_{i_m}, v_{j_m}), $1 \leq \ell, m \leq t$ can interchange positions in the ordered string without changing the value of $r_{s_k}(G^{\rightarrow})$. We say that s_k has the commutative property.

Clearly, pairs of predator arcs can be grouped together for preferred sequential application prior to other predator arcs, meaning $s_k = ((v_{i_1} \leadsto v_{j_1}), (v_{i_2} \leadsto v_{j_2}), (v_{i_3} \leadsto v_{j_3}), ..., (v_{i_t} \leadsto v_{j_t})) = (((v_{i_s} \leadsto v_{j_s}), (v_{i_\ell} \leadsto v_{j_\ell})), (v_{i_w} \leadsto v_{j_w})_{\forall other\ arcs, w \neq s, \ell})$. We say that s_k has the associative property. The next two lemmas follow.

Lemma 2.2. *Consider a specific orientation of a simple connected graph G on $n \geq 2$ vertices labelled $v_1, v_2, ..., v_n$ say, $G^{\rightarrow}$. For the initial cumulative predator$_{\geq 0}$-prey$_{\geq 0}$ population given by $\sum_{i=1}^{n} \rho(v_i) = \sum_{i=1}^{n} i$, we have that:*

$$
r_{s_k}(G^{\rightarrow})
\begin{cases}
= even, & \text{if and only if, } \sum_{i=1}^{n} i \text{ is even,} \\[2em]
= uneven, & \text{if and only if, } \sum_{i=1}^{n} i \text{ is uneven.}
\end{cases}
$$

Proof. Note that if vertex v_i predator along the arc (v_i, v_j) in step $*$ of the Grog algorithm then $\rho_*(v_i) = \rho_{*-1}(v_i) - 1$ and $\rho_*(v_j) = \rho_{*-1}(v_j) - 1$ so the total reduction is always 2 for each predator arc in $\mathbb{P}_{s_k}$. Hence, $2c(\mathbb{P}_{s_k}(G^{\rightarrow}))$ is always even.

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The two parts now follow immediately from Number Theory. $\qquad\square$

Lemma 2.3. *For a specific orientation of a simple connected graph G on $n \geq 2$ vertices labelled $v_1, v_2, ..., v_n$ say, $G^{\rightarrow}$ we have that $c(\mathbb{P}_{s_k}(G^{\rightarrow})) = \frac{1}{2}(\sum\limits_{i=1}^{n} i - r_{s_k}(G^{\rightarrow}))$.*

Proof. From Lemma 2.2 it follows that $r_{s_k}(G^{\rightarrow}) = \sum\limits_{i=1}^{n} i - 2c(\mathbb{P}_{s_k}(G^{\rightarrow}))$. Hence the result:

$$c(\mathbb{P}_{s_k}(G^{\rightarrow})) = \tfrac{1}{2}(\sum\limits_{i=1}^{n} i - r_{s_k}(G^{\rightarrow})). \qquad\square$$

2.2   On Paths and Cycles

Proposition 2.4. *If a path $P_n, n \geq 3$ and any specific orientation thereof say, $P_n^{\rightarrow}$ is extended to $P_{n+1}^{\rightarrow}$ the residual population over all possible predator-prey strategies applicable to $P_{n+1}^{\rightarrow}$ is given by:*

$$r_{s_k^*}(P_{n+1}^{\rightarrow}) \begin{cases} = r_{s_k}(P_n^{\rightarrow}) + (n+1), & \text{if and only if } v_s = v_1, \\[2mm] = r_{s_k}(P_n^{\rightarrow}) + (n-1), & \text{otherwise,} \end{cases}$$

and s_k^* is the minimal deviation from s_k to accommodate arcing to or from v_{n+1} and, v_{n+1} is linked to an end vertex v_s of $P_n^{\rightarrow}$ or;

$$r_{s_k^*}(P_{n+1}^{\rightarrow}) \begin{cases} = r_{s_k}(P_n^{\rightarrow}) + n, & \text{if and only if either } v_p = v_1 \text{ or } v_q = v_1, \\[2mm] = r_{s_k}(P_n^{\rightarrow}) + (n-1), & \text{otherwise,} \end{cases}$$

and s_k^* is the minimal deviation from s_k to accommodate arcing to or from v_{n+1} and, v_{n+1} squeesed inbetween two vertices v_p, v_q of $P_n^{\rightarrow}$ with $1 \leq p, q \leq n$.

Proof. Consider P_{n+1}. We begin by considering any specific orientation of P_n having end vertices $v_s, v_t, 1 \leq s, t \leq n$ and denote it $P_n^{\rightarrow}$. Clearly the extension from $P_n^{\rightarrow}$ to $P_{n+1}^{\rightarrow}$ is made possible by linking (arcing) vertex v_{n+1} to either v_s or v_t or *squeesing* it between two vertices of $P_n^{\rightarrow}$, say v_p, v_q with $1 \leq p, q \leq n$.

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Case 1: Assume v_{n+1} is linked to v_s.

Subcase 1.1: If we consider the arc (v_{n+1}, v_s) in $P_{n+1}^{\rightarrow}$ the one strategy $r_{s_k}^*$, could be for v_{n+1} to prey on v_s first, leaving a portion of the residual population amounting to n at v_{n+1} and a portion of the residual population amounting to $s - 1$ at v_s. However if $v_s = v_1$ the vertex v_1 cannot predator further on its neighbor in $P_n^{\rightarrow}$ anymore. So, if s_k is applied from vertex v_s throughout the rest of the remaining path then we have:

$$r_{s_k^*}(P_{n+1}^{\rightarrow}) \begin{cases} = r_{s_k}(P_n^{\rightarrow}) + (n+1), & \text{if and only if } v_s = v_1, \\ \\ = r_{s_k}(P_n^{\rightarrow}) + (n-1), & \text{otherwise.} \end{cases}$$

If we consider the strategy to apply s_k effecting from v_s first, the vertex v_{n+1} cannot predator at all if $v_s = v_1$. If $v_s \neq v_1$ there is a loss of 2 at v_s and a loss of 1 at v_{n+1} from the initial total $predator_{\geq 0}$-$prey_{\geq 0}$ population in $P_{n+1}^{\rightarrow}$. Hence the result:

$$r_{s_k^*}(P_{n+1}^{\rightarrow}) \begin{cases} = r_{s_k}(P_n^{\rightarrow}) + (n+1), & \text{if and only if } v_s = v_1, \\ \\ = r_{s_k}(P_n^{\rightarrow}) + (n-1), & \text{otherwise,} \end{cases}$$

holds.

Subcase 1.2: If we consider the arc (v_s, v_{n+1}) in $P_{n+1}^{\rightarrow}$ the one strategy $r_{s_k}^*$, could be for v_s to prey on v_{n+1} first, leaving a portion of the residual population amounting to n at v_{n+1} and a portion of the residual population amounting to $s - 1$ at v_s. Now, if s_k is applied from vertex v_s throughout the rest of the remaining path then we have:

$$r_{s_k^*}(P_{n+1}^{\rightarrow}) \begin{cases} = r_{s_k}(P_n^{\rightarrow}) + (n+1), & \text{if and only if } v_s = v_1, \\ \\ = r_{s_k}(P_n^{\rightarrow}) + (n-1), & \text{otherwise.} \end{cases}$$

If we consider the strategy to apply s_k effecting from v_s first, the vertex v_{n+1} cannot predator at all if $v_s = v_1$. If $v_s \neq v_1$ there is a loss of 2 at v_s and a loss of 1 at v_{n+1} from the initial total $predator_{\geq 0}$-$prey_{\geq 0}$ population in $P_{n+1}^{\rightarrow}$. Hence the result:

$$r_{s_k^*}(P_{n+1}^{\rightarrow}) \begin{cases} = r_{s_k}(P_n^{\rightarrow}) + (n+1), & \text{if and only if } v_s = v_1, \\ \\ = r_{s_k}(P_n^{\rightarrow}) + (n-1), & \text{otherwise,} \end{cases}$$

holds.

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Case 2: Assume v_{n+1} is linked to v_t.

The proof of this case follows similar to that of Case 1.

Case 3: Assume v_{n+1} is squeesed inbetween vertices v_p, v_q with $1 \leq p, q \leq n$.

Subcase 3.1: The arcs (v_p, v_{n+1}) and (v_{n+1}, v_q) exist in $P_{n+1}^{\rightarrow}$. The one strategy $r_{s_k}^*$ in $P_{n+1}^{\rightarrow}$, could be for v_p to prey on v_{n+1} first, leaving a portion of the residual population amounting to n at v_{n+1} and a portion of the residual population amounting to $p-1$ at v_p. Then let v_{n+1} prey on v_q, leaving a portion of the residual population amounting to $n-1$ at v_{n+1} and a portion of the residual population amounting to $q-1$ at v_q. Now, if s_k is applied throughout the rest of the remaining path then we have:

$$
r_{s_k^*}(P_{n+1}^{\rightarrow}) \begin{cases} = r_{s_k}(P_n^{\rightarrow}) + n, & \text{if and only if either } v_p = v_1 \text{ or } v_q = v_1, \\ \\ = r_{s_k}(P_n^{\rightarrow}) + (n-1), & \text{otherwise,} \end{cases}
$$

holds.

Subcase 3.2: The arcs (v_p, v_{n+1}) and (v_q, v_{n+1}) exist in $P_{n+1}^{\rightarrow}$. By applying the strategy s_k^* to accommodate vertex v_{n+1} and with similar reasoning as in Subcase 3.1, the result:

$$
r_{s_k^*}(P_{n+1}^{\rightarrow}) \begin{cases} = r_{s_k}(P_n^{\rightarrow}) + n, & \text{if and only if either } v_p = v_1 \text{ or } v_q = v_1, \\ \\ = r_{s_k}(P_n^{\rightarrow}) + (n-1), & \text{otherwise,} \end{cases}
$$

holds.

Subcase 3.3: The arcs (v_{n+1}, v_p) and (v_{n+1}, v_q) exist in $P_{n+1}^{\rightarrow}$. By applying the strategy s_k^* to accommodate vertex v_{n+1} and with similar reasoning as in Subcase 3.1, the result:

$$
r_{s_k^*}(P_{n+1}^{\rightarrow}) \begin{cases} = r_{s_k}(P_n^{\rightarrow}) + n, & \text{if and only if either } v_p = v_1 \text{ or } v_q = v_1, \\ \\ = r_{s_k}(P_n^{\rightarrow}) + (n-1), & \text{otherwise,} \end{cases}
$$

holds.

Subcase 3.4: The arcs (v_q, v_{n+1}) and (v_{n+1}, v_p) exist in $P_{n+1}^{\rightarrow}$. By applying the strategy s_k^* to

accommodate vertex v_{n+1} and with similar reasoning as in Subcase 3.1, the result:

$$r_{s_k^*}(P_{n+1}^{\rightarrow}) \begin{cases} = r_{s_k}(P_n^{\rightarrow}) + n, & \text{if and only if either } v_s = v_1 \text{ or } v_q = v_1, \\ \\ = r_{s_k}(P_n^{\rightarrow}) + (n-1), & \text{otherwise,} \end{cases}$$

holds. □

Corollary 2.5. *For a path* $P_n, n \geq 3$ *we have that* $g(P_{n+1}) = g(P_n) + (n-1)$.

Proof. From Proposition 2.4 it follows that for a specific orientation of P_n and for all possible predator-prey strategies,

$$g(P_{n+1}^{\rightarrow}) = min\{r_{s_k}(P_n^{\rightarrow}) + (n+1), r_{s_k}(P_n^{\rightarrow}) + n, r_{s_k}(P_n^{\rightarrow}) + (n-1)\}_{\forall s_k \in S(P_n^{\rightarrow})} =$$

$$min\{r_{s_k}(P_n^{\rightarrow})\}_{\forall s_k \in S(P_n^{\rightarrow})} + (n-1).$$

From definition 2.3 it then follows that $g(P_{n+1}) = g(P_n) + (n-1)$. □

Example 1. Consider path P_3. Note vertex labelling will be from *left to right*. Also note that the labelled vertices will be denoted through an *ordered triplet* and the arcs through an *ordered airc-pair*. The $\frac{1}{2}.3!.2^2 = 12$ distinct predator-prey webs are:

(1) $V(P_3^{\rightarrow}) = \{v_1, v_2, v_3\}$ and $A(P_3^{\rightarrow}) = \{(v_1, v_2), (v_2, v_3)\}$
(2) $V(P_3^{\rightarrow}) = \{v_1, v_2, v_3\}$ and $A(P_3^{\rightarrow}) = \{(v_1, v_2), (v_3, v_2)\}$
(3) $V(P_3^{\rightarrow}) = \{v_1, v_2, v_3\}$ and $A(P_3^{\rightarrow}) = \{(v_2, v_1), (v_2, v_3)\}$
(4) $V(P_3^{\rightarrow}) = \{v_1, v_3, v_2\}$ and $A(P_3^{\rightarrow}) = \{(v_1, v_3), (v_3, v_2)\}$
(5) $V(P_3^{\rightarrow}) = \{v_1, v_3, v_2\}$ and $A(P_3^{\rightarrow}) = \{(v_1, v_3), (v_2, v_3)\}$
(6) $V(P_3^{\rightarrow}) = \{v_1, v_3, v_2\}$ and $A(P_3^{\rightarrow}) = \{(v_3, v_1), (v_3, v_2)\}$
(7) $V(P_3^{\rightarrow}) = \{v_2, v_1, v_3\}$ and $A(P_3^{\rightarrow}) = \{(v_2, v_1), (v_1, v_3)\}$
(8) $V(P_3^{\rightarrow}) = \{v_2, v_1, v_3\}$ and $A(P_3^{\rightarrow}) = \{(v_2, v_1), (v_3, v_1)\}$
(9) $V(P_3^{\rightarrow}) = \{v_2, v_1, v_3\}$ and $A(P_3^{\rightarrow}) = \{(v_1, v_2), (v_1, v_3)\}$
(10) $V(P_3^{\rightarrow}) = \{v_2, v_3, v_1\}$ and $A(P_3^{\rightarrow}) = \{(v_2, v_3), (v_3, v_1)\}$
(11) $V(P_3^{\rightarrow}) = \{v_3, v_1, v_2\}$ and $A(P_3^{\rightarrow}) = \{(v_3, v_1), (v_1, v_2)\}$
(12) $V(P_3^{\rightarrow}) = \{v_3, v_2, v_1\}$ and $A(P_3^{\rightarrow}) = \{(v_3, v_2), (v_2, v_1)\}$

For each case the number of greedy strategies together with the residual population $r_{s_k}(P_3^{\rightarrow})$ as well as the grog number $g(P_3^{\rightarrow})$ will be depicted.

(1) Number of greedy strategies $= 2$; $r_{s_1}(P_3^{\rightarrow}) = 2$, $r_{s_2}(P_3^{\rightarrow}) = 2$ and $g(P_3^{\rightarrow}) = min\{2, 2\} = 2$.
(2) Number of greedy strategies $= 2$; $r_{s_1}(P_3^{\rightarrow}) = 2$, $r_{s_2}(P_3^{\rightarrow}) = 2$ and $g(P_3^{\rightarrow}) = min\{2, 2\} = 2$.
(3) Number of greedy strategies $= 2$; $r_{s_1}(P_3^{\rightarrow}) = 2$ $r_{s_2}(P_3^{\rightarrow}) = 2$ and $g(P_3^{\rightarrow}) = min\{2, 2\} = 2$.
(4) Number of greedy strategies $= 2$; $r_{s_1}(P_3^{\rightarrow}) = 2$, $r_{s_2}(P_3^{\rightarrow}) = 2$ and $g(P_3^{\rightarrow}) = min\{2, 2\} = 2$.
(5) Number of greedy strategies $= 2$; $r_{s_1}(P_3^{\rightarrow}) = 2$, $r_{s_2}(P_3^{\rightarrow}) = 2$ and $g(P_3^{\rightarrow}) = min\{2, 2\} = 2$.
(6) Number of greedy strategies $= 2$; $r_{s_1}(P_3^{\rightarrow}) = 2$, $r_{s_2}(P_3^{\rightarrow}) = 2$ and $g(P_3^{\rightarrow}) = min\{2, 2\} = 2$.
(7) Number of greedy strategies $= 2$; $r_{s_1}(P_3^{\rightarrow}) = 4$, $r_{s_2}(P_3^{\rightarrow}) = 4$ and $g(P_3^{\rightarrow}) = min\{4, 4\} = 4$.
(8) Number of greedy strategies $= 2$; $r_{s_1}(P_3^{\rightarrow}) = 4$, $r_{s_2}(P_3^{\rightarrow}) = 4$ and $g(P_3^{\rightarrow}) = min\{4, 4\} = 4$.
(9) Number of greedy strategies $= 2$; $r_{s_1}(P_3^{\rightarrow}) = 4$, $r_{s_2}(P_3^{\rightarrow}) = 4$ and $g(P_3^{\rightarrow}) = min\{4, 4\} = 4$.
(10) Number of greedy strategies $= 2$; $r_{s_1}(P_3^{\rightarrow}) = 2$, $r_{s_2}(P_3^{\rightarrow}) = 2$ and
$g(P_3^{\rightarrow}) = min\{2, 2\} = 2$.
(11) Number of greedy strategies $= 2$; $r_{s_1}(P_3^{\rightarrow}) = 4$, $r_{s_2}(P_3^{\rightarrow}) = 4$ and
$g(P_3^{\rightarrow}) = min\{4, 4\} = 4$.
(12) Number of greedy strategies $= 2$; $r_{s_1}(P_3^{\rightarrow}) = 2$, $r_{s_2}(P_3^{\rightarrow}) = 2$ and
$g(P_3^{\rightarrow}) = min\{2, 2\} = 2$.

From definition 2.3 it follows that $g(P_3) = min\{2, 4\} = 2$.

Theorem 2.6. *For all simple connected graphs on $n \geq 3, n \in \mathbb{N}$ vertices, over all indices and over all orientations of G, there exists at least one indice with at least one orientation say orientation o_i with corresponding directed graph $G_{o_i}^{\rightarrow}$ and at least another indice with at least one orientation say orientation o_j with corresponding directed graph $G_{o_j}^{\rightarrow}$, $o_i \neq o_j$ with $g(G_{o_i}^{\rightarrow}) \neq g(G_{o_j}^{\rightarrow})$.*

Proof. In any simple connected graph G on $n \geq 3, n \in \mathbb{N}$, at least one induced subgraph $\langle G \rangle_i$ on $i \leq n$ vertices exists with $\langle G \rangle_i \simeq P_3$. Hence in any simple connected graph G on $n \geq 3$, $n \in \mathbb{N}$ vertices we can find as a minimal indiced case, the subgraph P_3 with $V(P_3) = \{v_1, v_2, v_3\}$.

Consider the orientation $o_1 = \{(v_1, v_2), (v_2, v_3)\}$. So for $P_{3, o_1}^{\rightarrow}$ we have $r_{s_1}(P_{3, o_1}^{\rightarrow}) = 2, r_{s_2}(P_{3, o_1}^{\rightarrow}) = 2$ and $g(P_{3, o_1}^{\rightarrow}) = min\{2, 2\} = 2$.

Now consider the orientation $o_2 = \{(v_2, v_1), (v_1, v_3)\}$. So for $P_{3, o_2}^{\rightarrow}$ we have $r_{s_1}(P_{3, o_2}^{\rightarrow}) = 4$, $r_{s_2}(P_{3, o_2}^{\rightarrow}) = 4$ and $g(P_{3, o_2}^{\rightarrow}) = min\{4, 4\} = 4$.

So we have $g(P_{3, o_1}^{\rightarrow}) \neq g(P_{3, o_2}^{\rightarrow})$.

Considering the partial graph $G - P_3$, let $g((G - P_3)_{o_i}^{\rightarrow} = t$ for a specific orientation o_i

and a *predator-prey strategy* s_k.

For the path $P_3 = v_1 v_2 v_3$ consider the graph $G^{\rightarrow}$ with the *predator-prey strategy* $s_k^* = (s_1, s_k)$ and the orientation $o_i^* = \{(v_1, v_2), (v_2, v_3), o_i\}$. Clearly arcs to and from $P_3^{\rightarrow}$ and the rest of $G^{\rightarrow}$ may exist such that upon applying s_k^* we have $g((G - P_3)_{o_i}^{\rightarrow}) - 4 = t - 4 \leq g(G^{\rightarrow})_{o_i^*} \leq t + 2 = g((G - P_3)_{o_i}^{\rightarrow}) + 2$.

For the path $P_3 = v_2 v_1 v_3$ consider the graph $G^{\rightarrow}$ with the *predator-prey strategy* $s_k^* = (s_1, s_k)$ and the orientation $o_j^* = \{(v_2, v_1), (v_1, v_3), o_j\}$. Clearly arcs to and from $P_3^{\rightarrow}$ and the rest of $G^{\rightarrow}$ may exist such that upon applying s_k^* we have $g((G - P_3)_{o_j}^{\rightarrow}) - 8 = t - 8 \leq g(G^{\rightarrow})_{o_j^*} \leq t + 4 = g((G - P_3)_{o_j}^{\rightarrow}) + 4$.

Since the set of arcs between the two directed paths and the rest of $G^{\rightarrow}$ must specifically remain the same, it follows that:

$$g((G - P_3)_{o_i}^{\rightarrow}) - 4 = t - 4 \leq g(G^{\rightarrow})_{o_i^*} \leq t + 2 = g((G - P_3)_{o_i}^{\rightarrow}) + 2 \neq$$

$$g((G - P_3)_{o_j}^{\rightarrow}) - 8 = t - 8 \leq g(G^{\rightarrow})_{o_j^*} \leq t + 4 = g((G - P_3)_{o_j}^{\rightarrow}) + 4.$$

Therefore, in general the result that there exist at least two different orientations and at least two different indices of G, such that $g(G_{o_i}^{\rightarrow}) \neq g(G_{o_j}^{\rightarrow})$, follows. $\square$

Proposition 2.7. *If a cycle $C_n, n \geq 3$ and any specific orientation thereof say, $C_n^{\rightarrow}$ is extended to $C_{n+1}^{\rightarrow}$ the residual population over all possible predator-prey strategies applicable to $C_{n+1}^{\rightarrow}$ is given by:*

$$r_{s_k^*}(C_{n+1}^{\rightarrow}) = r_{s_k}(C_n^{\rightarrow}) + (n - 1),$$

and s_k^* is the minimal deviation from s_k to accommodate arcing to or from v_{n+1}.

Proof. Consider C_{n+1}. We begin by considering any specific orientation of C_n and denote it $C_n^{\rightarrow}$. Clearly the extension from $C_n^{\rightarrow}$ to $C_{n+1}^{\rightarrow}$ is made possible by *squeesing* v_{n+1}, (arcing) inbetween two vertices of $C_n^{\rightarrow}$, say v_s, v_t with $1 \leq s, t \leq n$.

Without loss of generality consider the arc (v_s, v_t) in $C_n^{\rightarrow}$. Begin by letting v_s prey on v_t by predatoring along the arc (v_s, v_t). After this first step of the Grog algorithm the rest of the application applies to a path $P_{n*}^{\rightarrow}$ with end vertices v_s and v_t having $\rho(v_s) = s - 1$ and $\rho(v_t) = t - 1$. After applying s_k to this path the value $r_{s_k}(C_n^{\rightarrow})$ is obtained.

We now squeese v_{n+1} inbetween v_s, v_t and for any one of the four possible orientation between v_s, v_{n+1}, v_t we have that if the Grog algorithm is applied to path $P_3^{\rightarrow}$, $V(P_3) = \{v_s, v_{n+1}, v_t\}$ we obtain $\rho(v_s) = s - 1$, $\rho(v_{n+1}) = n - 1$, $\rho(v_t) = t - 1$.

Furthering with the Grog algorithm we are left with exactly the path $P_{n*}^{\rightarrow}$ mentioned above. Hence the result:

$$r_{s_k^*}(C_{n+1}^{\rightarrow}) = r_{s_k}(C_n^{\rightarrow}) + (n - 1),$$

and s_k^* is the minimal deviation from s_k to accommodate v_{n+1}, follows. $\qquad\square$

Corollary 2.8. *For a cycle $C_n, n \geq 3$ we have that $r_{s_k^*}(C_n^{\rightarrow}) = r_{s_k}(P_n^{\rightarrow}) - 2$, with s_k^* the minimal deviation from s_k to accommodate an orientation of the edge $v_p v_q$ with v_p and v_q the end vertices of $P_n^{\rightarrow}$.*

Proof. The result follows directly from the proof of Proposition 2.7. $\qquad\square$

2.3   On Jaco graphs, $J_n(1), n \in \mathbb{N},\ n \geq 2$

As stated in Kok et. al. [3], finding a closed formula for the number of edges of a finite Jaco graph will assist in finding closed formulae for many recursive results found for Jaco graphs. In the absence of such formula we present the next proposition. We begin with a lemma.

Lemma 2.9. *For a Jaco graph, $J_n(1)$, $n \geq 2$ having the Jaconian vertex v_i we have that $2i - n \geq 0$.*

Proof. From the definition of a Jaco graph $J_n(1)$, $n \geq 2$ with Jaconian vertex v_i we have that either $i + d^+(v_i) = n$ or $i + d^+(v_i) = n - 1$. Therefore:

Case 1: $i + d^+(v_i) = n$

$\therefore i = n - d^+(v_i)$,
$\therefore 2i = 2n - 2d^+(v_i)$,
$\therefore 2i - n = 2n - 2d^+(v_i) - n = n - 2d^+(v_i) = (n - d^+(v_i)) - d^+(v_i) = i - d^+(v_i)$.

Since $i - d^+(v_i) = d^-(v_i)$ and $d^-(v_i) \geq 0$ in $J_n(1)$, $n \geq 2$ the result follows.

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Case 2: $i + d^+(v_i) = n - 1$

$\therefore i = (n-1) - d^+(v_i),$
$\therefore 2i = 2(n-1) - 2d^+(v_i),$
$\therefore 2i - n = 2(n-1) - 2d^+(v_i) - n = n - 2d^+(v_i) - 2 = ((n-1) - d^+(v_i)) - d^+(v_i) - 1$
$= (i - d^+(v_i)) - 1 = d^-(v_i) - 1.$

Since $d^-(v_i) \geq 1$ in $J_n(1)$, $n \geq 2$ the result follows. $\qquad\square$

Proposition 2.10. *For a Jaco graph, $J_n(1)$, $n \geq 2$ having the Jaconian vertex v_i we have that:*

$$g(J_{n+1}(1)) = g(J_n(1)) + (2i - n) + 1.$$

Proof. Consider any Jaco graph $J_n(1)$, $n \geq 2$ with Jaconian vertex v_i and grog number $g(J_n(1))$. In extending to $J_{n+1}(1)$ the vertex v_{n+1} with arcs $(v_{i+1}, v_{n+1}), (v_{i+2}, v_{n+1}), ..., (v_n, v_{n+1})$ are added to $J_n(1)$. So clearly $n - i$ additional arcs are added.

By applying the Grog algorithm to vertices v_j, $i + 1 \leq j \leq n$ along the respective arcs (v_j, v_{n+1}), $i + 1 \leq j \leq n$ there is a corresponding cumulative reduction in the residual population at vertices v_j, $i + 1 \leq j \leq n$ of $n - i$. Furthermore, there is an corresponding increase in the residual population at vertex v_{n+1} of $(n + 1) - (n - i)$. Hence,

$$g(J_{n+1}(1)) = g(J_n(1)) - (n - i) + ((n + 1) - (n - i)) = g(J_n(1)) + (2i - n) + 1. \qquad\square$$

Corollary 2.11. *For a Jaco graph, $J_n(1)$, $n \geq 2$ we have $g(J_{n+1}(1)) > g(J_n(1))$.*

Proof. Since $(2i - n) \geq 0$, $n \geq 3$ (Lemma 2.9) and $g(J_n(1)) + 1 > g(J_n(1))$, $n \geq 2$ it follows that $g(J_n(1)) + (2i - n) + 1 > g(J_n(1))$, $n \geq 2$. Therefore, $g(J_{n+1}(1)) > g(J_n(1))$, $n \geq 2$. $\qquad\square$

[Open problem 1: Formalise Observation 1, mathematically.]
[Open problem 2: Formalise Observation 2, mathematically.]
[Open problem 3: The Grog algorithm has been described informally. Formalise the Grog algorithm.]
[Open problem 4: For a given $G^{\rightarrow}$ find the number of possible predator-prey strategies, (cardinality of $S(G^{\rightarrow})$) and if possible describe the algorithmic efficiency of determining it.]

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